

# Symmetry and Self-Duality in Categories of Probabilistic Models

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This note adds to the recent spate of derivations of the probabilistic apparatus of finite-dimensional quantum theory from various axiomatic packages. We offer two different axiomatic packages that lead easily to the Jordan algebraic structure of finite-dimensional quantum theory. The derivation relies on the Koecher-Vinberg Theorem, which sets up an equivalence between order-unit spaces having homogeneous, self-dual cones, and formally real Jordan algebras.

## 1 Introduction and Overview

The last several years have seen a spate of derivations of the probabilistic apparatus of finite-dimensional quantum theory from various axiomatic packages, many having an information-theoretic motivation [7, 8, 10, 12, 13]. This note (which in part echoes, but greatly improves upon [19]) adds to the flow. I offer two different, though overlapping, axiomatic packages, both stressing symmetry principles, that lead quickly and easily to the Jordan algebraic structure of finite-dimensional quantum theory. Quickly and easily, at any rate, if one is familiar with the Koecher-Vinberg Theorem [11, 15], which sets up an equivalence between order-unit spaces having homogeneous, self-dual cones, and formally real Jordan algebras.

A probabilistic system can be described, in a standard way, in terms of an order-unit space  $A$ , the positive elements of which are scalar multiples of “effects”. The strategy, then, is to show that certain strong, but not unreasonable, assumptions force the positive cone  $A_+$  to be homogeneous and self-dual, and hence, isomorphic to the cone of squares of such a Jordan algebra. In [3], several conditions are adduced that lead to a homogeneous and *weakly* self-dual cone — that is, a homogeneous cone that is *order-isomorphic* to its dual cone in  $A^*$ . However, proper self-duality is a much more stringent condition, requiring that the isomorphism be mediated by an inner product.

The line of attack here is to assume that systems individually have a great deal of symmetry, and collectively, can be organized into a symmetric monoidal category [1, 2, 14]. Here is a sketch. Further details can be found in the longer paper [20].

## 2 Probabilistic Models

A *test space* [16] is a pair  $(X, \mathfrak{A})$  where  $X$  is a set of *outcomes* and  $\mathfrak{A}$  is a covering of  $X$  by non-empty (for our purposes here, finite) subsets called *tests*, each understood as the set of possible outcomes of some measurement, experiment, etc. Two outcomes  $x, y \in X$  are *distinguishable* iff they belong to a common test. In this case, I write  $x \perp y$ . Notice that there is, as yet, no linear structure in view, let alone an inner product; so this notation is promissory.

A *state* on a test space  $(X, \mathfrak{A})$  is a function  $\alpha : X \rightarrow [0, 1]$ , summing independently to unity on each test. A *symmetry* of  $(X, \mathfrak{A})$  is a mapping  $g : X \rightarrow X$  such that  $g(E), g^{-1}(E) \in \mathfrak{A}$  for every  $E \in \mathfrak{A}$ . By a *probabilistic model*, I mean a structure  $(X, \mathfrak{A}, \Omega, G)$  where  $(X, \mathfrak{A})$  is a test space,  $\Omega$  is a compact convex set of states on  $(X, \mathfrak{A})$ , and  $G$  is a group acting on  $(X, \mathfrak{A})$  by symmetries, and leaving  $\Omega$  invariant.

For illustration, if  $\mathbf{H}$  is a finite-dimensional Hilbert space (real or Complex), the corresponding *quantum model* is  $(X(\mathbf{H}), \mathfrak{A}(\mathbf{H}), \Omega(\mathbf{H}), U(\mathbf{H}))$ , where  $X = X(\mathbf{H})$  is the set of rank-one projection operators on  $\mathbf{H}$ ,  $\mathfrak{A} = \mathfrak{A}(\mathbf{H})$  is set of (projective) *frames*, i.e., maximal pairwise orthogonal sets of projections,  $\Omega(\mathbf{H})$  is the convex set of density operators on  $\mathbf{H}$ , and  $U(\mathbf{H})$  is the group of unitary operators on  $\mathbf{H}$ , acting on  $X(\mathbf{H})$  by conjugation.

**Categories of Models.** I will be interested in categories of models. A *morphism* from a model  $(X, \mathfrak{A}, \Omega, G)$  to a model  $(Y, \mathfrak{B}, \Gamma, H)$  is a pair  $(\phi, \psi)$ , where

- (i)  $\phi : X \rightarrow Y$  with  $\phi(\mathfrak{A}) \subseteq \mathfrak{B}$ ,  $\phi^*(\Gamma) \subseteq \Omega$
- (ii)  $\psi \in \text{Hom}(G, H)$ ;
- (iii)  $\phi(gx) = \psi(g)\phi(x)$  for all  $x \in X, g \in G$ .

In what follows,  $\mathcal{C}$  is a symmetric monoidal category of probabilistic models  $A = (X(A), \mathfrak{A}(A), \Omega(A), G(A))$ , with morphisms as above. I shall make two further assumptions:

- (1) For every  $A \in \mathcal{C}$ ,  $G(A) \subseteq \mathcal{C}(A, A)$ .
- (2) The model  $A \otimes B \in \mathcal{C}$  is a *composite* of the models  $A, B \in \mathcal{C}$ , in the sense of [5]. This means, in particular, that there are canonical injections  $\otimes : X(A) \times X(B) \rightarrow X(A \otimes B)$  and  $\otimes : \Omega(A) \times \Omega(B) \rightarrow \Omega(A \otimes B)$ , with

$$E \otimes F = \{x \otimes y | x \in E, y \in F\} \in \mathfrak{A}(A \otimes B)$$

for every  $E \in \mathfrak{A}(A), F \in \mathfrak{A}(B)$ , and

$$(\alpha \otimes \beta)(x \otimes y) = \alpha(x)\beta(y)$$

for all  $\alpha \in \Omega(A), \beta \in \Omega(B), x \in X(A)$  and  $y \in X(B)$ . A *bipartite state* between  $A, B \in \mathcal{C}$  is a state  $\omega$  in  $\Omega(A \otimes B)$ . It is also part of the definition that the *un-normalized conditional state*  $\hat{\omega}(x) := \omega(x, \cdot)$  belong to  $\Omega(B)$  for every  $x \in X$ , and similarly with  $A$  and  $B$  reversed.

**Models Linearized.** Every model  $A = (X(A), \mathfrak{A}(A), \Omega(A), G(A)) \in \mathcal{C}$  generates, in a standard and quite canonical way, an order-unit space  $\mathbf{E}(A)$ . To be precise,  $\mathbf{E}(A)$  is the span in  $\mathbb{R}^\Omega$  of the evaluation functionals associated with measurement outcomes  $x \in X$ . In the case of a quantum model  $A(\mathbf{H}) = (X(\mathbf{H}), \mathfrak{A}(\mathbf{H}), \Omega(\mathbf{H}), U(\mathbf{H}))$ , one has  $\mathbf{E}(A) \simeq \mathcal{L}(\mathbf{H})$ , the space of Hermitian operators on  $\mathbf{H}$ , with the usual ordering and  $u_A = \mathbf{1}_\mathbf{H}$ .

The construction  $A \mapsto \mathbf{E}(A)$  is functorial, so we obtain from  $\mathcal{C}$  a category  $\mathbf{E}(\mathcal{C})$  of order-unit spaces and positive linear mappings. It is natural to enlarge this to a category  $\mathcal{E}$  in which each hom-set  $\mathcal{E}(A, B)$  is an ordered linear space, and in which, e.g.,  $\mathcal{E}(I, A) \simeq \mathbf{E}(A)$ . In what follows, I assume that such a “linearized” category  $\mathcal{E}$  has been fixed.

### 3 Bi-Symmetric Models

To tighten this structure further, I now ask that every  $A \in \mathcal{C}$  enjoy a property I call *bi-symmetry*.

**Definition.** A model  $A \in \mathcal{C}$  is *bi-symmetric* iff

- (i)  $G(A)$  acts transitively on the pure states (that is, extreme points) of  $\Omega(A)$ ,
- (ii)  $G(A)$  acts transitively on  $\mathfrak{A}$ , and on pairs  $(x, y)$  of outcomes with  $x \perp y$ .

If, in place of (ii), we require that arbitrary bijections  $f : E \rightarrow F$ ,  $E, F \in \mathfrak{A}$ , extend to elements of  $G$ , then  $A$  is *fully bi-symmetric*.

If  $A$  is bi-symmetric, then  $G$  acts transitively. Clearly, the quantum model discussed above is fully bi-symmetric. Bi-symmetry and full bi-symmetry, are very natural conditions. (See [17] for further discussion and motivation of the latter.)

**Definition.** A *SPIN form*<sup>1</sup> for the model  $A$  is a real bilinear form  $B$  on  $\mathbf{E}(A)$  that is *symmetric*, *positive* in the sense that  $B(a, b) \geq 0$  for all  $a, b \in \mathbf{E}(A)_+$ , *invariant*, in the sense that  $B(ga, gb) = B(a, b)$  for all  $g \in G(A)$ , and *normalized*, in the sense that  $B(u_A, u_A) = 1$ . A SPIN form is *orthogonalizing* iff  $B(x, y) = 0$  for all distinguishable measurement outcomes  $x, y \in X(A)$ .

An example is the usual tracial inner product on  $\mathcal{L}(\mathbf{H})$ . Call  $\mathbf{E}(A)$  *irreducible* iff (with respect to any SPIN form  $B$ ), the subspace  $u^\perp = \{a \in \mathbf{E}(A) \mid B(a, u) = 0\}$  (this is independent of  $B$ ) is irreducible under the group  $G(A)$ . Quantum models are irreducible in this sense.

**Theorem 1.** *If  $\mathbf{E}(A)$  is irreducible, it supports at most one orthogonalizing SPIN form, which — if it exists — is an inner product.*

## 4 Conjugates and Daggers

At this point, the aim is to find sufficient conditions for the existence of an orthogonalizing SPIN form on  $\mathbf{E}(A)$ . I shall provide two.

**Definition.** By a *conjugate* for a model  $A$ , I mean a structure  $(\bar{A}, \gamma_A, \eta_A)$ , where  $\gamma_A : A \mapsto \bar{A}$  is an isomorphism of models, and  $\eta_A$  is a bipartite state on  $A \times \bar{A}$  such that  $\eta(x, \gamma_A(x)) = 1/n$  (where  $n$  is the rank of  $A$ ) for every  $x \in X(A)$ .

In the case of a quantum-mechanical model  $A = A(\mathbf{H})$  associated with a Hilbert space  $\mathbf{H}$ , the obvious conjugate model is just that associated with the conjugate Hilbert space  $\bar{\mathbf{H}}$ , with  $\gamma_A$  taking the rank-one projection  $x$  to the corresponding projection  $\bar{x}$  on  $\bar{\mathbf{H}}$ , and with  $\eta_A$  the pure state associated with the unit vector  $\frac{1}{\sqrt{n}} \sum_i e_i \otimes \bar{e}_i$ ,  $\{e_i\}$  any basis for  $\mathbf{H}$  (note that this is basis-independent).

Returning to the general case, note that by averaging over  $G(A)$ , we can choose  $\eta_A$  to be invariant, in the sense that  $\eta_A(gx, \gamma_A(gy)) = \eta_A(gx, gy)$  for all  $g \in G(A)$ . This gives us an invariant SPIN form on  $\mathbf{E}(A)$ , defined on outcomes by  $B(x, y) := \eta(x, \gamma_A(y))$ . Applying Theorem 1, we have

**Theorem 2.** *Let  $\mathbf{E}(A)$  be irreducible, and suppose  $A$  has a conjugate. Then  $\mathbf{E}(A)$  carries a canonical orthogonalizing inner product.*

Under some mild auxiliary hypotheses, the existence of a conjugate for every  $A \in \mathcal{C}$  (with  $\gamma_A$  and  $\eta_A$  appropriately belonging to  $\mathcal{C}$ ) can be used to construct a dagger on the category  $\mathcal{E} \supseteq \mathbf{E}(\mathcal{C})$  discussed above. In fact, however, the mere existence of a reasonable dagger-monoidal structure on  $\mathcal{E}$  is enough to obtain much the same result.

**Theorem 3.** *Suppose  $\mathcal{E}$  supports a dagger-monoidal structure, such that for every  $g \in G(A)$ ,  $g^\dagger = g^{-1}$  (i.e.,  $g \in G(A)$  is “unitary”). If  $\mathbf{E}(A)$  is irreducible, then it carries an orthogonalizing inner product.*

In order to obtain the self-duality of  $\mathbf{E}(A)_+$  for an irreducible model  $A$ , it now suffices to assume either of two simple further conditions:

**Theorem 4.** *Suppose that either*

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<sup>1</sup>This is probably not the best choice of terminology.

- (a) In the context of Theorem 2,  $A$  has a conjugate such that the state  $\eta_A$  is an isomorphism state or
- (b) In the context of Theorem 3,  $A$  is sharp, meaning that every outcome has probability one in a unique state on  $\mathbf{E}(A)$ .

Then  $\mathbf{E}(A)_+$  is self-dual.

The homogeneity of  $\mathbf{E}(A)_+$  can now be enforced by any of several conditions discussed in [4, 19]. Applying the Koecher-Vinberg Theorem, we can conclude that  $\mathbf{E}(A)$  carries a unique Jordan product making it a formally real Jordan algebra.

One of these conditions is so simple it's worth pausing to describe it. Any bipartite state  $\omega$  between  $A, B \in \mathcal{C}$  gives rise to a natural positive linear mapping  $\hat{\omega} : \mathbf{E}(A) \rightarrow \mathbf{E}(B)^*$ , uniquely defined by  $\hat{\omega}(x)(y) = \omega(x, y)$ . Where  $\hat{\omega}$  is an *order-isomorphism* — that is, where  $\hat{\omega}$  is an order-isomorphism (that is, invertible and having a positive inverse), we call  $\omega$  an *isomorphism state*. A basic observation from [4], translated into the present context, is that if every state in the interior of  $\Omega(B)$  is the marginal of an isomorphism state, then the cone in  $\mathbf{E}(B)$  generated by  $\Omega(B)$  is homogeneous.

## 5 Image-Closure

In order to extend these results to possibly reducible systems, I impose one further constraint on  $\mathcal{C}$ . Call a morphism  $(\phi, \psi) : (X, \mathfrak{A}, \Omega, G) \rightarrow (Y, \mathfrak{B}, \Gamma, H)$  is *surjective* iff  $\phi(X) = Y$ ,  $\mathfrak{B} \subseteq \phi(\mathfrak{A})$ ,  $H = \psi(G)$ , and  $\Gamma = \{\beta \in \Omega(Y, \mathfrak{B}) \mid \phi^*(\beta) \in \Omega\}$ . In this case, we call  $(Y, \mathfrak{B}, \Gamma, H)$  the *image* of  $(X, \mathfrak{A}, \Omega, G)$  under  $(\phi, \psi)$ . Call  $\mathcal{C}$  *image-closed* iff, for any  $A \in \mathcal{C}$  and any surjective morphism  $(\phi, \psi) : (X_A, \mathfrak{A}_A, \Omega_A, G_A) \rightarrow (Y, \mathfrak{B}, \Gamma, H)$ , (i) the model  $B := (Y, \mathfrak{B}, \Gamma, H)$  belongs to  $\mathcal{C}$ , and (ii)  $(\phi, \psi) \in \mathcal{C}(A, B)$ . in  $\mathcal{C}$ , again belong to  $\mathcal{C}$ .

**Theorem 5.** *Let  $\mathcal{C}$  be an image-closed category of bi-symmetric probabilistic models, and let  $\mathcal{E}$  be the corresponding linearized category as discussed in Section 1. If either*

- (a) *every  $A \in \mathcal{C}$  has a conjugate  $\bar{A} \in \mathcal{C}$ , with  $\eta_A$  an isomorphism state, or*
- (b)  *$\mathcal{E}$  has a dagger-monoidal structure making every  $g \in G(A)$  unitary for all  $A \in \mathcal{C}$ , and every  $A \in \mathcal{C}$  is sharp, then for every  $A \in \mathcal{C}$ ,  $\mathbf{E}(A)_+$  is self-dual.*

Again, adding any of the sufficient conditions for homogeneity from [4, 17] — or simply assuming it outright — will yield a category of formally real Jordan algebras.

Operationally, it is reasonable to suppose that any image  $\phi(A)$  of a model  $A \in \mathcal{C}$  can be *simulated* by means of the model  $A$ . Hence, if we wish to think of  $\mathcal{C}$  as closed under operationally reasonable constructions, it is not far-fetched that  $\phi(A)$  should belong to  $\mathcal{C}$ . In fact, the image of a bi-symmetric model is 2-symmetric, so one can simply “close up”  $\mathcal{C}$  without sacrificing this assumption. (To suppose that this closure continues to support, e.g., a symmetric-monoidal structure, or conjugate systems, is a sharper constraint, of course.) Categories of finite-dimensional quantum models turn out to be image-closed for the simple reason that a quantum model *has no* non-trivial images.

## 6 Conclusion

These results raise any number of interesting questions. For one thing, it is possible that the assumptions are stronger than advertised, singling out a narrower class than formally real Jordan algebras. It is noteworthy that I have not had to assume that  $\mathcal{C}$ 's monoidal product is locally tomographic. In fact, in a forthcoming paper with Howard Barnum [6], we show (using a result of Hanche-Olsen) that if  $\mathcal{C}$  is

a dagger-monoidal category of finite-dimensional order-unit spaces with homogeneous self-dual cones, then local tomography, plus the existence in  $\mathcal{C}$  of a system having the structure of a qubit, implies that every  $A \in \mathcal{C}$  is isomorphic to the Hermitian part of a finite-dimensional complex  $C^*$  algebra.

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